Comparison properties of stochastic decision free Petri nets

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Abstract

This paper focuses on stochastic comparison problems arising in the analysis of Stochastic Decision Free Petri Nets (SDFPN). The analysis is based on the evolution equations satisfied by firing times. Various structural properties are obtained including the association of the firing times and their stochastic and convex monotonicity with respect to the holding time sequences and to the initial marking. The association and the stochastic monotonicity properties are extended to the counters using an inversion relation. It is also proved that the counters and the throughput are stochastically concave in the initial marking provided the holding times are i.i.d and belong to a subclass of log-concave distribution functions that is introduced in the paper. Various bounds for the asymptotic cycle time and the throughput are then derived from these stochastic ordering results. Finally, results are presented concerning the stochastic and convex monotonicity of the transient and the stationary marking distributions.

Keywords: Discrete Event Dynamic System, Stochastic Decision Free Petri Nets, Firing Time, Counter, Cycle Time, Throughput, Marking Distribution, Upper and Lower Bounds, Association, Stochastic Ordering, Convex Ordering, Log Concavity.

Propriétés de comparaison stochastique dans les graphes d'événements aléatoires

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Résumé

Cet article est consacré aux propriétés de monotonie et de convexité stochastique vérifiées par les graphes d'événements aléatoires. L'analyse est fondée sur les équations d'évolution vérifiées par les dates de franchissement. Les premiers résultats concernent l'association des dates de franchissement, leur monotónie et leur convexité stochastique par rapport aux séquences des durées de franchissement et par rapport au marquage initial. Les propriétés d'association et la monotonie stochastique sont ensuite étendues aux compteurs d'événements en utilisant la relation d'inversion qui les lie aux dates de franchissement. Nous démontrons aussi que les compteurs et le débit sont stochastiquement concaves par rapport au marquage initial, dans le cas où les durées de franchissement sont indépendantes et identiquement distribuées et appartiennent à une classe de variables aléatoires de fonction de répartition log-concave qui est introduite dans l'article. Plusieurs bornes calculables sur le temps de cycle et le débit du graphe d'événement sont déduites de ces résultats.

Keywords: Systèmes à événements discrets, graphes d'événements stochastiques, dates de franchissement, compteur, temps de cycle, débit, marquage initial, association, ordonnancement stochastique, convexité stochastique, log-concavité.

1 Introduction

This paper focuses on stochastic comparison problems arising in the analysis of Stochastic Decision Free Petri Nets (SDFPN). The derivation of our results is based on the evolution equations satisfied by firing times that were established by Baccelli in [5].

The SDFPNs under consideration consist of a subclass of stochastic Petri nets, and are also called marked graphs or event graphs in the literature. A brief description of such networks together with the basic stochastic ordering concepts to be used in the paper are provided in §2. Section 3 contains all the stochastic monotonicity stochastic convexity and association properties satisfied by firing times, which represent the epochs when certain events occur in the network. The corresponding properties for counters, which count the number of events of a certain kind that occurred before a given time, are given in §4. This section also contains a proof of the stochastic concavity of the counters in the initial marking, provided the holding times are i.i.d and belong to a subclass of log-concave distribution functions called PERT distribution functions (see §4 below), which includes as special cases Erlangian distributions. Section 5 focuses on the cycle time and the throughput of such networks, for which various bounds and convexity properties are derived using the transient stochastic monotonicity properties of §3 and §4. The marking distribution and the response times of the tokens are analyzed in §6, where stochastic and convex monotonicity properties are derived.

Everywhere throughout the paper, we stress how the Petri net approach that is proposed here unifies various and apparently unrelated results of the queueing literature in a few general theorems, and in particular known results on queues in tandem, blocking queues, fork-join queues, synchronized queueing networks etc. Several new queueing theory results can also be obtained, for instance by applying our results to the general Fork/Join queueing networks with blocking introduced in Ammar and Gershwin [3] which were shown to be equivalent to strongly connected SDFPN's by Dallery, Liu and Towsley [16]. However, we will not pursue this line of thoughts as the main purpose of the paper is to show that the Petri net setting provides a unifying framework, and not to give all the possible queueing interpretations of these stochastic monotonicity results.

2 Notation and Definitions

2.1 Model Description

The basic model of this paper is a Stochastic Decision Free Petri Net with recycled transitions. We assume that tokens incur no sojourn times in places and that there is at most one place between two transitions. The definition of this class of Petri net is sketched below (see [5] for more details on the matter).

- $\mathcal{T} = \{1, \ldots, J\}$: the set of transitions;
- $\pi(\cdot)$: the predecessor function ($\pi(j)$ is the set of transitions preceding j);
- $\sigma(\cdot)$: the successor function ($\sigma(j)$ the set of transitions that follow transition j);
- $\Gamma = (\mathcal{V}, \mathcal{E})$: the directed graph defined by the precedence relation π on the set $\mathcal{V} = \mathcal{T}$;
- $\mathcal{P}$: the set of places. Each place is preceded and followed by exactly one transition (this is the so called decision free property). There is a place between j and j' iff $(j, j') \in \mathcal{E}$; this place will be denoted (j, j');
- $\mu(j, j') \in I\!N$: the initial marking in place $(j, j') \in \mathcal{P}$. It is assumed by convention that $\mu(j, j') = \infty$ if $(j, j') \notin \mathcal{P}$.
- All the transitions are *recycled*, vz., for all $1 \leq j \leq J$, $j \in \pi(j)$ and the place (j, j) has an initial marking $\mu(j, j) = 1$.
- $M = \max_{(j,j') \in \mathcal{P}} \mu(j, j')$ is the maximum initial marking;
- $\alpha_j(k) \in I\!R^+$ is the holding time of the k-th firing of transition $j \in \mathcal{T}$, $k \geq 1$.

The evolution of the SDFPN is characterized by the circulation of *tokens*, which stay in places, and are consumed and created by transitions. A transition j is enabled to *fire* when there is at least one token in each of the places (i, j), $i \in \pi(j)$. The firing consumes one token of each of these places and creates, after some holding time $\alpha_j(k)$, $k \geq 1$, one token into each of the places (j, j'), $j' \in \sigma(j)$. We assume that the firing of a transition, takes place as soon as it is enabled. In the literature, SDFPN's are also called *marked graphs* ([6,15]) or *event graphs* ([5,14]).

Without loss of generality, we can assume that the SDFPN is connected. Moreover, in order to guarantee the *liveness* of the SDFPN (i.e., each transition fires infinitely many times), we assume that for each cycle in the graph Γ, there is at least one place with a positive initial marking ([15]).

2.2 Statistical Assumptions

Throughput this paper, we assume that the sequences $\{\alpha_j(k)\}_{k=1}^{+\infty}$, $j = 1, \ldots, J$, are non-negative and integrable RV's (random variables) defined on a common probability space

(Ω, F, P).

2.3 Notions on Integral Orderings, Association and Stochastic Concavity

Let $\mathcal{D}(I\!R^n)$ denote the space of distribution functions on $I\!R^n$ and $\mathcal{L}$ be a set of borel mappings from $I\!R^n$ into $I\!R$. Consider the partial ordering $\leq_{\mathcal{L}}$ on $\mathcal{D}(I\!R^n)$ defined by

$$F \leq_{\mathcal{L}} \tilde{F} \qquad \text{iff} \qquad \int_{I\!R^n} f(x)F(dx) \leq \int_{I\!R^n} f(x)\tilde{F}(dx),$$

for all $f \in \mathcal{L}$ such that the integrals are well defined. The partial orderings considered in this paper are the basic stochastic ordering (generated by the set of coordinatewise nondecreasing functions), the convex ordering (generated by the set of convex functions), and the increasing convex ordering (generated by the set of convex and nondecreasing functions), respectively denoted $\leq_{st}$, $\leq_{cx}$, and $\leq_{icx}$ (see [28] for various properties and applications of stochastic orderings).

Let

$$\{x(k)\}_k = \{x(1), \ldots, x(k), \ldots\}$$

and

$$\{\tilde{x}(k)\}_k = \{\tilde{x}(1), \ldots, \tilde{x}(k), \ldots\}$$

be two $I\!R^K$-valued stochastic sequences on the probability space $(\Omega, \mathcal{F}, P)$. The sequence $\{\tilde{x}(k)\}_k$ will be said to dominate $\{x(k)\}_k$ for the $\leq_{\mathcal{L}}$ ordering, which will be denoted

$$\{x(k)\}_k \leq_{\mathcal{L}} \{\tilde{x}(k)\}_k,$$

if for all $n \geq 1$, the distribution functions F and $\tilde{F}$ defined by $F(u) = P[x(1) \leq u_1, \ldots, x(n) \leq u_n]$ and $\tilde{F}(u) = P[\tilde{x}(1) \leq u_1, \ldots, \tilde{x}(n) \leq u_n]$, where $u_k \in I\!R^K$, satisfy the ordering relation $F \leq_{\mathcal{L}} \tilde{F}$ in $\mathcal{D}(I\!R^{K \times n})$.

We will also use the notion of association of random variables: the real valued random variables $x_1, \ldots, x_k$ are said to be associated if

$$cov[h(x_1, \ldots, x_n), g(x_1, \ldots, x_n)] \geq 0,$$

for all pairs of increasing functions $h, g : I\!R^n \to R$ such that the integral is well defined (see [13] and [12] for a survey on the matter).

A collection $\{Z(\theta), \theta \in \Theta\}$ of random variables with a convex parameter set $\Theta \subset I\!R^m$ is said to be strongly stochastically increasing and concave in θ if $E(\phi(Z(\theta)))$ is nondecreasing and concave in $\theta \in \Theta$ for all nondecreasing functions ϕ (see [24] for properties and applications of strong stochastic concavity).

3 Properties of Firing Times

3.1 Evolution Equations

In this subsection, we summarize results that were obtained in [5]. Let $X_j(k)$ denote the time when transition j starts firing for the k-th time. Whenever the initial condition is zero, these variables satisfy the evolution equation

$$X_j(k) \quad = \quad \max\left(\max_{\{i \in \pi(j)\}} (X_i(k-\mu(i,j)) + \alpha_i(k-\mu(i,j))), 0\right), \quad k = 1,2,\ldots, \tag{3.1}$$

where, by convention, the maximum over an empty set is $-\infty$, and $X_j(k) = -\infty$ if $k \leq 0$.

Since the decision free net is live, the numbering of the transitions can be chosen in such a way that for all (j,k), $j = 1\ldots, J$, $k \geq 1$, the variables $X_{j'}(k')$ that are found in the R.H.S of (3.1) are always such that either $k' < k$ or $k' = k$, $j' < j$. Therefore, the state variables $X_j(k)$ can be computed recursively in the order

$$X_1(1), X_2(1), \ldots, X_J(1), X_1(2), X_2(2), \ldots, X_J(2), \ldots, X_1(k), X_2(k), \ldots, X_J(k), \ldots. \tag{3.2}$$

Consider the semi-ring $(I\!R, \oplus, \otimes)$, where $\oplus$ is max and $\otimes$ is $+$. If the SDFPN under consideration is live, it is possible to rewrite this equation in matrix form in this semi-ring

$$X(k) = X(k-M) \otimes A(k-M,k) \oplus \cdots \oplus X(k-1) \otimes A(k-1,k) \oplus 0, \; k = 1,2,\ldots. \tag{3.3}$$

In this equation, the vector $X(k)$ is $(X_1(k), \ldots, X_J(k))$, for $k \geq 1$, $(-\infty, \ldots, -\infty)$, for $k \leq 0$. The vector 0 is $(0, \ldots, 0)$. The matrix $A(k-l,k)$ is defined as follows: let $\mathcal{S}(j',j,l)$ be the set of paths in the graph Γ with at least two edges, with initial vertex $q_{j'}$ and final vertex q_j, and such that the first two transitions of the path are connected by a place with initial marking equal to l, while the other transitions are connected by places with zero initial marking. Then

$$A_{j',j}(k-l,k) = \bigoplus_{\{\ (j'=i_0, i_1, j_2, i_2 \ldots, i_{h-1}, i_h = j)\ \in \mathcal{S}(j',j,l)\}} \alpha_{j'}(k-l) \otimes \left(\bigotimes_{m=1}^{h-1} \alpha_{i_m}(k)\right), \tag{3.4}$$

with the usual convention if the set $\mathcal{S}(j',j,l)$ is empty. The entry $A_{j',j}(k-l,k)$ is hence simply the length of the longest path in $\mathcal{S}(j',j,l)$.

3.2 Stochastic Monotonicity

3.2.1 Stochastic Monotonicity with respect to Holding Times

In the sequel, we use for $\alpha(k)$ the same vector notations as above, namely $\alpha(k) = (\alpha_1(k), \ldots, \alpha_J(k))$. ¿From the matrices $A(k, k-l)$, we define the $M \times J \times J$ dimensional vectors $A(k) = (A_{i,j}(k-M,k), \ldots, A_{i,j}(k-1,k),\ i,j = 1,\ldots,J)$. Let $\{\tilde{\alpha}(k)\}_{k=0}^{\infty}$ be another set of holding time sequences, and let $\{\widetilde{X}(k)\}_{k=0}^{\infty}$ be the state variables obtained for the same decision free Petri net as above, but with $\{\alpha(k)\}_k$ replaced by $\{\tilde{\alpha}(k)\}_k$. The aim of this section is to compare the stochastic processes $\{\widetilde{X}(k)\}_k$ and $\{X(k)\}_k$, whenever the sequences $\{\alpha(k)\}_k$ and $\{\tilde{\alpha}(k)\}_k$ compare for some integral ordering.

Proposition 3.1 *If*

$$\{\alpha(k)\}_k \le_{icx} \{\tilde{\alpha}(k)\}_k, \tag{3.5}$$

then

$$\{X(k)\}_k \le_{icx} \{\widetilde{X}(k)\}_k. \tag{3.6}$$

Proof. Observe first that $\{A(k)\}_k$ is a nondecreasing and convex function of $\{\alpha(k)\}_k$ (cf. (3.4)). Therefore (3.5) implies

$$\{A(k)\}_k \le_{icx} \{\tilde{A}(k)\}_k. \tag{3.7}$$

This theorem is then an immediate corollary of Theorem 2.5.2, Ch. 4, in [8] (see also §3 in [12]), and of the fact that $X(k)$ satisfies Equation (3.3), which can be put under the form

$$(X(k+1), \ldots, X(k-M+2)) = \Phi((X(k), \ldots, X(k-M+1)), A(k)), \tag{3.8}$$

where Φ is nondecreasing and convex. ∎

Various classical results that can be seen as corollaries of this result. We provide some examples in the following remarks.

Remark 3.1 In the deterministic version of the SDFPN Γ, namely the same decision free Petri net with holding times sequences $\bar{\alpha}_j(k) = E[\alpha_j(k)]$, the sequence of firing times $\{\bar{X}(k)\}_k$ that is obtained for this timing is a lower bound of $\{X(k)\}_k$ in the $\le_{icx}$ sense (see [12]). This idea was also used in [9] by the authors for deriving computable lower bounds of response times of parallel programs in multiprocessor systems.

Remark 3.2 Applying Proposition 3.1 to the cyclic queueing network with finite buffers yields a result of Shanthikumar and Yao [26] on departure times.

Remark 3.3 This result also allows one to extend the validity of Ross' conjecture to this class of Petri nets: if the holding times are doubly stochastic in the sense defined in [12], then the state variables associated with the firing times are bounded from below (in the convex ordering sense) by the firing times of the same system in a fixed environment equal to the mean environment.

Proposition 3.2 *If*

$$\{\alpha(k)\}_k \leq_{st} \{\tilde{\alpha}(k)\}_k, \tag{3.9}$$

then

$$\{X(k)\}_k \leq_{st} \{\tilde{X}(k)\}_k. \tag{3.10}$$

Proof. Same as above with Theorem 2.5.1, Ch. 4 in [8] in place of 2.5.1. ∎

3.2.2 Stochastic Monotonicity with respect to the Initial Marking

Consider the same SDFPN as in §2, but with the initial marking $\mu'(i,j)$, $i,j = 1,\ldots,J$, in place of $\mu(i,j)$. Let $\{X'(k)\}_k$ denote the corresponding firing times.

Proposition 3.3 *If*

$$\mu_i \leq \mu'_i, \quad i = 1,\ldots,J, \tag{3.11}$$

then

$$\{X(k)\}_k \geq_{st} \{X'(k)\}_k. \tag{3.12}$$

Proof. The proof is based on Equation (3.1). We prove the pathwise bound $X_j(k) \leq X'_j(k)$ for all (j,k). The proof is by induction on (j,k), based on the total ordering indicated in (3.2). Assume the property holds up to (j,k) at the exclusion of this point (it holds for $k \leq 0$).

Owing to the assumption that all transitions are recycled, it is easy to check that

$$X_j(k) \geq X_j(k-1) + \alpha_j(k-1), \quad \forall j = 1,\ldots,J,\ k \in \mathbb{Z}. \tag{3.13}$$

Therefore

$$X_j(k) \geq X_j(k-1), \quad \forall j = 1,\ldots,J,\ k \in \mathbb{Z}, \tag{3.14}$$

and

$$X_j(k) + \alpha_j(k) \geq X_j(k-1) + \alpha_j(k-1), \quad \forall j = 1,\ldots,J, \; k \in \mathbb{Z}. \tag{3.15}$$

Therefore, we have

$$\begin{aligned} X'_j(k) &= \max\left(\max_{\{i\in\pi(j)\}} (X'_i(k-\mu'(i,j)) + \alpha_i(k-\mu'(i,j))), 0\right) \\ &\leq \max\left(\max_{\{i\in\pi(j)\}} (X'_i(k-\mu(i,j)) + \alpha_i(k-\mu(i,j))), 0\right) \\ &\leq \max\left(\max_{\{i\in\pi(j)\}} (X_i(k-\mu(i,j)) + \alpha_i(k-\mu(i,j))), 0\right) \\ &= X_j(k), \end{aligned}$$

where we successively used the monotonicity property (3.15), and the induction assumption. ∎

Remark 3.4 As an application of this result, one gets the stochastic ($\leq_{st}$) monotonicity of departure times in closed cyclic networks as a function of the population. More generally, the stochastic ($\leq_{st}$) monotonicity of departure times holds for Fork/Join queueing networks with manufacturing or communication blocking [3,16] as a function of the buffer sizes and/or the population.

3.2.3 Stochastic Monotonicity with respect to the Topology

Consider a SDFPN with associated graph $\bar{\Gamma} = ((\bar{P} \cup \bar{T}), \bar{E})$, where

$$\mathcal{P} \subseteq \bar{\mathcal{P}}, \qquad \mathcal{T} \subseteq \bar{\mathcal{T}}, \qquad \mathcal{E} \subseteq \bar{\mathcal{E}}. \tag{3.16}$$

This new SDFPN is such that the initial marking and the holding times of those places and transitions that belong both to Γ and $\bar{\Gamma}$ are the same. Assume that $\bar{\Gamma}$ satisfies the assumptions of SDFPN in §2. Let $\{\bar{X}(k)\}_k$ denote the firing times of $\bar{\Gamma}$.

Proposition 3.4 *Under the foregoing assumptions,*

$$\{X(k)\}_k \leq_{st} \{\bar{X}(k)\}_k. \tag{3.17}$$

Proof. The proof is based on Equation (3.1). We prove the pathwise bound $\bar{X}_j(k) \leq X_j(k)$ for all (j,k). The proof is by induction on (j,k). Assume the property holds up to

(j,k) at the exclusion of this point (it holds for $k \leq 0$). Then for all $j \in \mathcal{V}$,

$$\begin{aligned}
\bar{X}_j(k) &= \max\left(\max_{\{i \in \bar{\pi}(j)\ \}} (\bar{X}_i(k - \bar{\mu}(i,j)) + \bar{\alpha}_i(k - \bar{\mu}(i,j))), 0 \right) \\
&\geq \max\left(\max_{\{i \in \pi(j)\}} (\bar{X}_i(k - \bar{\mu}(i,j)) + \bar{\alpha}_i(k - \bar{\mu}(i,j))), 0 \right) \\
&= \max\left(\max_{\{i \in \pi(j)\}} (\bar{X}_i(k - \mu(i,j)) + \alpha_i(k - \mu(i,j))), 0 \right) \\
&\geq \max\left(\max_{\{i \in \pi(j)\}} (X_i(k - \mu(i,j)) + \alpha_i(k - \mu(i,j))), 0 \right) \\
&= X_j(k),
\end{aligned}$$

where we successively used the assumptions (3.16), the assumption that the initial marking and the holding times of the vertices of $\mathcal{V} \cap \bar{\mathcal{V}}$ are the same and the induction assumption. ■

Remark 3.5 In [9] the authors obtained computable upper and lower bounds (in $\leq_{st}$ sense) for response times in synchronized queueing networks by adding and removing synchronization constraints, respectively. These results are a special case of the preceding proposition.

Remark 3.6 Similarly, using the preceding proposition, one can get upper and lower bounds (in $\leq_{st}$ sense) for the departure times in Fork/Join queueing networks with blocking (such as the model considered in [3]) by adding or removing synchronization or blocking constraints.

3.3 Association

Proposition 3.5 *If*

$$\{\alpha_j(k),\ j = 1, \ldots, J,\ k \geq 0\}$$

is a set of associated random variables, then

$$\{X_j(k),\ j = 1, \ldots, J,\ k \geq 0\}$$

is also a set of associated random variables.

Proof. It is easy to prove that the association assumption on the α's implies that the random variables $\{A(k, k-l)_{ij},\ i,j = 1, \ldots, J,\ k \in \mathbb{Z}, l = 1, \ldots, M\}$ are also associated. The property follows immediately from the result of III.D.2 in [12], applied to Equation (3.8). ∎

Since independent random variables are associated, the firing times are associated whenever the holding times are mutually independent.

4 Properties of Counters

4.1 Evolution Equations

Let $N_j(t)$ (resp. $Q_j(t)$) denote the number of firings that transition j initiates (resp. completes) by time t. Without loss of generality, we assume that both $N_j(t)$ and $Q_j(t)$ are right continuous. It will also be assumed that the holding times are all strictly positive.

Proposition 4.1 *The random variables $N_j(t)$ and $Q_j(t)$, $1 \leq j \leq J$, $t \geq 0$, satisfy the following evolution equations:*

$$N_j(t) = \min_{\{i \in \pi(j)\}} (Q_i(t) + \mu(i,j)), \tag{4.1}$$

$$Q_j(t) = \int_0^t 1_{\{\alpha_j(N_j(u)) < t-u\}} dN_j(u), \tag{4.2}$$

where $Q_j(0) = 0$, for all $j = 1, \ldots, J$.

Proof By time t, transition j initiates exactly as many firings as the minimum over $i \in \pi(j)$ of the number of tokens that entered place (i,j) by time t (including the initial tokens). Since a place is preceded by exactly one transition, the number of tokens that entered place (i,j) by t equals $\mu(i,j)$ plus the number of firings completed by transition i by time t. ∎

Corollary 4.2 *If the holding times are deterministic ($\alpha_j(k) = \alpha_j$), then these variables satisfy the equation*

$$N_j(t) = \min_{\{i \in \pi(j)\}} (N_i(t - \alpha_i) + \mu(i,j)), \tag{4.3}$$

where $N_i(t) = 0$ for $t \leq 0$.

Remarks Observe that $X_j(k)$ and $N_j(t)$ are 'inverse' processes related by the formulas

$$N_j(t) = \sum_{k=1}^{\infty} 1_{\{X_j(k) \leq t\}} = \inf\{k \mid X_j(k+1) > t\}, \tag{4.4}$$

$$X_j(k) = \inf\{t \mid N_j(t) \geq k\}. \tag{4.5}$$

It then follows that

$$\begin{aligned}
N_j(t) &= \inf\{k \mid X_j(k+1) > t\} \\
&= \inf\{k \mid \exists j_1 : \ X_{j_1}(k+1-\mu(j_1,j)) > t - \alpha_{j_1}(k+1-\mu(j_1,j))\} \\
&= \inf\{k \mid \exists j_1, j_2 : \ X_{j_2}(k+1-\mu(j_1,j)-\mu(j_2,j_1)) > \\
&\quad t - \alpha_{j_1}(k+1-\mu(j_1,j)) - \alpha_{j_2}(k+1-\mu(j_2,j_1)-\mu(j_1,j))\} \\
&\quad \ldots \\
&= \inf\{k \mid \exists j_1, j_2, \ldots, j_n : \ X_{j_n}(k+1-\mu(j_1,j)-\mu(j_2,j_1)-\cdots-\mu(j_n,j_{n-1})) > \\
&\quad t - \alpha_{j_1}(k+1-\mu(j_1,j)) - \alpha_{j_2}(k+1-\mu(j_2,j_1)-\mu(j_1,j)) \\
&\quad - \ldots - \alpha_{j_n}(k+1-\mu(j_n,j_{n-1})-\cdots-\mu(j_1,j))\}
\end{aligned} \tag{4.6}$$

Consider the case where the initial marking is at most 1 (it is always possible to find an equivalent network with this property). Since for all $i \in \mathcal{T}$ and $k < 0$, $X_i(k) = -\infty$, there is no need to consider sequences $j_1, \ldots, j_n$ such that $k+1-\mu(j_1,j)-\mu(j_2,j_1)-\cdots-\mu(j_n,j_{n-1}) \leq 0$ in the last expression. Therefore, for all k as above, there exists $j_1, \ldots, j_n$, such that

$$k+1-\mu(j_1,j)-\mu(j_2,j_1)-\cdots-\mu(j_n,j_{n-1}) = 1, \quad \text{and} \quad X_{j_n}(1) = 0, \tag{4.7}$$

where the property $X_{j_n}(1) = 0$ comes from the fact that for all i, either $X_i(1) = 0$, or there exist transitions $i = i_1, i_2, \ldots, i_m$ such that $i_{u+1} \in \pi(i_u)$ and $\mu(i_{u+1}, i_u) = 0$, $1 \leq u \leq m-1$ and $X_{i_m}(1) = 0$. Substituting (4.7) in (4.6) implies that

$$\begin{aligned}
N_j(t) &= \inf\{\mu(j_n,j_{n-1}) + \cdots + \mu(j_2,j_1) + \mu(j_1,j) \mid \\
&\quad \alpha_{j_n}(1) + \alpha_{j_{n-1}}(1+\mu(j_n,j_{n-1})) + \cdots \\
&\quad + \alpha_{j_1}(1+\mu(j_n,j_{n-1})+\cdots+\mu(j_2,j_1)+\mu(j_1,j)) > t\}
\end{aligned} \tag{4.8}$$

∎

4.2 Stochastic Monotonicity and Association

One can see from (4.4) that for any fixed $t \geq 0$, $N_j(t)$ is a nonincreasing function of $\{X_j(k)\}_k$. Therefore, each $\leq_{st}$-monotonicity property of the processes $X_j(k)$ with respect to $\leq_{st}$ yields a dual stochastic monotonicity property of $N_j(t)$. In particular, we have the following immediate corollaries of the propositions of §3.2:

Corollary 4.3 *Under the assumptions of Proposition 3.2 (monotonicity with respect to holding times in the $\leq_{st}$ sense),*

$$\{(N_1(t), \ldots, N_J(t))\}_t \geq_{st} \{(\tilde{N}_1(t), \ldots, \tilde{N}_J(t))\}_t. \tag{4.9}$$

Corollary 4.4 *Under the assumptions of Proposition 3.3 (monotonicity with respect to the initial marking),*

$$\{(N_1(t), \ldots, N_J(t))\}_t \leq_{st} \{(N'_1(t), \ldots, N'_J(t))\}_t. \tag{4.10}$$

Corollary 4.5 *Under the assumption of Proposition 3.4 (monotonicity with respect to the topology),*

$$\{(N_1(t), \ldots, N_J(t))\}_t \geq_{st} \{(\bar{N}_1(t), \ldots, \bar{N}_J(t))\}_t. \tag{4.11}$$

In the same vein, we have

Corollary 4.6 *If*

$$\{\alpha_j(k),\ j = 1, \ldots, J,\ k \geq 0\}$$

is a set of associated random variables, then, for all $t_1, t_2, \ldots, t_n$ in $I\!R^+$,

$$\{N_j(t_i),\ j = 1, \ldots, J,\ i = 1, \ldots, n\}$$

is also a set of associated random variables.

Proof. Owing to Proposition 3.5 and Equation (4.4), one obtains that $\{X_j(k),\ j = 1, \ldots, J,\ k \geq 0, -N_j(t_i),\ j = 1, \ldots, J,\ i = 1, \ldots, n$ is a set of associated random variables. ∎

Remark 4.1 Applying Corollary 4.4 yields the monotonicity result of Adan and van der Wal [1] for counters with respect to buffer sizes in an assembly network.

Remark 4.2 Applying Corollaries 4.3—4.5 to the Fork/Join queueing networks with blocking implies the monotonicity of the counters with respect to the holding times, the initial marking and the topology.

4.3 Concavity with respect to the Initial Marking

Throughout this subsection, it is assumed that the sequences $\{\alpha_j(k)\}_k$ are mutually independent in j.

Proposition 4.7 *If the random variables $\alpha_j(k)$ are i.i.d., with exponential distribution of parameter λ_j, then, for any $t \geq 0$, and any $1 \leq j \leq J$, $N_j(t)$ and $Q_j(t)$ are strongly stochastically increasing and concave in the initial marking $\mu \in I\!N^{|\mathcal{E}|}$.*

Proof. Let $\{\beta_j(n)\}_{n=1}^{\infty}$, $1 \leq j \leq J$, be mutually independent sequences of i.i.d. random variables where $\beta_j(n)$ is exponentially distributed with parameter λ_j. Let $T_0, T_1, T_2, \ldots, T_n, \ldots$ be the times defined by $T_0 = 0$ and

$$T_n \;=\; T_{n-1} + \min_{1 \leq j \leq J} \beta_j(n), \qquad n \geq 1, \tag{4.12}$$

and let $U_j(n)$ be the indicator function

$$U_j(n) = 1_{\{T_n = T_{n-1} + \beta_j(n)\}}. \tag{4.13}$$

Let Γ' be another SDFPN with the same topology and initial marking as Γ, and with the following dynamics: in Γ', for all transitions j enabled at time T_n^+, take the residual firing time in j at time T_n^+ to be $\beta_j(n+1)$. If $U_j(n+1) = 1$, transition j fires at time T_{n+1}^-, which defines a the set of enabled transitions at time T_{n+1}^+. If $U_{j'}(n+1) = 1$ for a transition j' that is not enabled, nothing happens. For each transition j that is enabled at T_{n+1}^+, take a new residual firing time equal to $\beta_j(n+2)$ etc.

For Γ' defined as above, it is not difficult to verify that the state variable $N_j'(t)$ (resp. $Q_j'(t)$) representing the number of firings of transition j initiated (resp. completed) by time t satisfies the following equations:

$$\begin{aligned}
Q_j'(0) &= 0, \\
Q_j'(t) &= Q_j'(T_n) = Q_j^n, \qquad T_n \leq t < T_{n+1}, \quad n = 0, 1, 2, \ldots, \\
N_j'(t) &= N_j'(T_n) = N_j^n, \qquad T_n \leq t < T_{n+1}, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{4.14}$$

where

$$N_j^n \;=\; \min_{i \in \pi(j)} \left(Q_i^n + \mu(i,j)\right), \tag{4.15}$$

$$Q_i^{n+1} \;=\; \min\left((Q_i^n + U_j(n+1)),\; N_i^n\right), \qquad n = 0, 1, 2, \ldots. \tag{4.16}$$

Equation (4.15) is obtained in the same way as (4.1). In order to get Equation (4.16) observe that

$$Q_j^{n+1} \leq Q_j^n + U_j(n+1),$$

(the equality holds if j is enabled at time T_n^+), and

$$Q_j^{n+1} \leq N_j^n,$$

(because of the recycling of transition j, there is at most one firing initiated and not completed). Equation (4.16) follows when using the property that j is enabled at T_n^+ iff $Q_j^n < N_j^n$.

It is now immediate to prove by induction that N_j^n and Q_j^n, $1 \leq j \leq J$, $n \geq 0$, are a nondecreasing and concave functions of μ. So are the variables $N_j'(t)$ and $Q_j'(t)$ in view of (4.14), and of the fact that the variables T_n do not depend upon μ.

Owing to the memoryless property of the exponential distribution, one can see that the state variables $N_j'(t)$ and $Q_j'(t)$, $1 \leq j \leq J$, are identical in distribution with $N_j(t)$ and $Q_j(t)$, respectively. Therefore $N_j(t)$ and $Q_j(t)$ are strongly stochastically increasing and concave in μ. ∎

We now define the class of PERT type distributions.

Definition 4.8 *A stochastic PERT graph is a weighted directed acyclic graph where the weights are random variables associated with vertices. The weight of the critical path of the stochastic PERT graph is the maximum of the weight of all the paths in the graph.*

Note that in the preceding definition, we assume that only the vertices are weighted, not the edges. There is no loss of generality in this assumption since one can equivalently weight the edges or both the edges and the vertices.

Definition 4.9 *The distribution function of a random variable X is of PERT type if X can be expressed as the weight of the critical path of a stochastic PERT graph G where the weights of the vertices are mutually independent and have exponential distributions. Such a distribution function will be denoted $F(G, \lambda_1, \ldots, \lambda_{|G|})$, where $\lambda_1, \ldots, \lambda_{|G|}$ are the parameters of the exponential distributions of the vertices of G.*

Definition 4.10 *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is log concave if for all $x, y \in \mathbb{R}^n$, the inequality $f(\alpha x + (1-\alpha)y) \geq f^\alpha(x) f^{(1-\alpha)}(y)$ holds for all $0 < \alpha < 1$.*

Proposition 4.11 *PERT type distribution functions are log concave.*

Proof. See Appendix A. ∎

Proposition 4.12 *If the holding times of a SDFPN are all mutually independent, and if for all transition j, $1 \leq j \leq J$, the holding times of j are i.i.d. random variables with PERT distribution function, $F_j(G_j, \lambda_{j,1}, \ldots, \lambda_{j,n_j})$, where $n_j = |G_j|$, then, for any $t \geq 0$, and any $1 \leq j \leq J$, $N_j(t)$ and $Q_j(t)$ are strongly stochastically increasing and concave in the initial marking μ.*

Proof. Assume first that there are a single source vertex (an source vertex is a vertex with no predecessor) and a single sink vertex (with no successor) in each PERT graphs G_j, $1 \leq j \leq J$. Let the source and the sink vertices of G_i be labeled 1 and n_j, respectively.

For all j, construct a stochastic Petri net Γ_j from the PERT graph G_j by replacing each vertex of G_j by a transition and by inserting a place on each edge connecting two transitions. The initial marking of each of these places is zero. The transition obtained from vertex i is labeled j_i, and its holding time is a RV with exponential distribution with parameter $\lambda_{j,i}$.

For all j, $1 \leq j \leq J$, replace transition j in the SDFPN Γ by the SDFPN associated with Γ_j, in such a way that $\pi(j_1) = \pi(j)$ and $\sigma(j_{n_j}) = \sigma(j)$. Furthermore, add an edge from transition j_{n_j} to transition j_1 with a place with one token in its initial marking. Let Γ' denote the SDFPN thus obtained.

It is readily checked that the two SDFPN's have the same time behavior. More precisely, it can be shown by an immediate coupling argument that for all j, $1 \leq j \leq J$,

$$N_j(t) =_{st} N'_{j_1}(t), \qquad Q_j(t) =_{st} Q'_{j_1}(t), \qquad t \geq 0,$$

where the symbol $=_{st}$ denotes the equivalence in law and where $N'_{j_1}(t)$ and $Q'_{j_1}(t)$ respectively denote the number of firings of transitions $j_1 \in \Gamma'$ initiated and completed by time t.

Let μ' denote the initial marking of Γ'. Applying Proposition 4.7 to Γ' entails that for any $t \geq 0$, and any $1 \leq j \leq J$, $N'_{j_1}(t)$ and $Q'_{j_1}(t)$ are strongly stochastically increasing and concave in the initial marking $\mu' \in I\!N^{|\mathcal{E}'|}$. Consequently, $N_j(t)$ and $Q_j(t)$ are strongly stochastically increasing and concave in the initial marking $\mu \in I\!N^{|\mathcal{E}|}$, $t \geq 0$, $1 \leq j \leq J$.

Consider now the case where the PERT graphs have multiple source and sink vertices. In principle, we can come back to the initial case by adding an extra source vertex that is a predecessor of all source vertices in G and with zero weight, and in doing the same for the sink. However, a problem arises as 0 is not a permissible weight within our framework (indeed, exponential distributions with an infinite parameter could not be handled in the uniformization method of Proposition 4.7). We overcome this difficulty by the following construction: for each G_j, $1 \leq j \leq J$, construct graphs G_j^n, $n \geq 1$, by inserting two vertices 0 and $|G_j| + 1$. The vertex 0 is the predecessor of all the source vertices of G_j, and the vertex $|G_j| + 1$ is the successor of all the sink vertices of G_j. The vertices 0 and $|G_j| + 1$ in G_j^n are weighted with independent random variables with exponential distributions of parameters $\lambda_{j,0}^n = n$ and $\lambda_{j,|G_j|+1}^n = n$, respectively.

Denote by Γ^n the SDFPN which has the same structure as Γ, but where the holding times of transition j, $\alpha_j^n(k)$, are i.i.d. random variables with PERT distribution function $F_j^n(G_j^n, \lambda_{j,0}^n, \lambda_{j,1}, \ldots, \lambda_{j,|G_j|}, \lambda_{j,|G_j|+1}^n)$, $1 \leq j \leq J$, $k \geq 1$. Denote by $N_j^n(t)$ and $Q_j^n(t)$ the counters of Γ^n. It follows from the preceding proof that for any $t \geq 0$, and any $1 \leq j \leq J$, $E[f(N_j^n(t))]$ and $E[f(Q_j^n(t))]$ are concave functions in μ, provided that f is nondecreasing.

According to Strassen's Theorem ([29], [19]) there exists a probability space in which the holding times of the SDFPN's under consideration satisfy the bounds

$$\alpha_j^n(k) \geq \alpha_j^{n+1}(k), \quad a.s., \qquad 1 \leq j \leq J, \quad k \geq 1.$$

Since $N_j^n(t)$ and $Q_j^n(t)$ are monotone functions of holding times (Corollary 4.3), the monotone convergence theorem implies

$$\lim_{n\to\infty} E[f(N_j^n(t))] = E\left[\lim_{n\to\infty} f(N_j^n(t))\right] = E[f(N_j^\infty(t))],$$

and

$$\lim_{n\to\infty} E[f(Q_j^n(t))] = E\left[\lim_{n\to\infty} f(Q_j^n(t))\right] = E[f(Q_j^\infty(t))].$$

Note that Γ^∞ have the same time behavior has Γ in the sense that

$$N_j^\infty(t) =_{st} N_j(t), \quad \text{and} \quad Q_j^\infty(t) =_{st} Q_j(t).$$

Since the concavity is preserved under the limit, $E[f(N_j(t))]$ and $E[f(Q_j(t))]$ are concave functions in μ, whenever f is nondecreasing. Differently stated, $N_j(t)$ and $Q_j(t)$ are strongly stochastically increasing and concave in the initial marking μ. ∎

Remark 4.3 It is easy to see that PERT type distributions include Erlang distributions as a special case. Therefore one can approximate step functions with PERT distributions. Thus, Proposition 4.12 still holds when the holding times are deterministic.

Remark 4.4 Proposition 4.7 generalizes the strong stochastic concavity result of Meester and Shanthikumar [21] and Anantharam and Tsoucas [4] on tandem queueing networks with finite buffers.

Remark 4.5 As an application, the strong stochastic concavity holds for general Fork/Join networks with blocking and with PERT type distributions.

5 Properties of Cycle Time and Throughput

Throughout this section, we will assume that the sequences of holding times $\{\alpha_j(k)\}_k$, $1 \le j \le J$, are jointly stationary and ergodic.

Let $X^*(k) = \max_{1\le j\le J} X_j(k)$, $k = 1, 2, \ldots$. Using subbaditive ergodic theory, it was shown in [5] that there exists a constant χ such that

$$\lim_{k\to\infty} X^*(k)/k = \lim_{k\to\infty} E[X^*(k)]/k = \chi \qquad a.s.. \tag{5.1}$$

Let $N^*(t) = \min_{1\le j\le J} N_j(t)$, $t \ge 0$. It is readily checked that

$$N^*(t) = \inf\{k \mid X^*(k+1) > t\}.$$

Hence for all $k \ge 1$,

$$N^*(t) = k, \qquad X^*(k) \le t < X^*(k+1),$$

which implies that

$$\frac{n}{X^*(n+1)} \le \frac{N^*(t)}{t} < \frac{n}{X^*(n)}, \qquad X^*(k) \le t < X^*(k+1).$$

Using now (5.1) yields

$$\lim_{t\to\infty} N^*(t)/t = \lim_{t\to\infty} E[N^*(t)]/t = \chi^{-1} \qquad a.s.. \tag{5.2}$$

The constants χ and χ^{-1} are referred to as the *asymptotic cycle time* (or simply cycle time) and the *throughput* of the SDFPN Γ.

In case of deterministic SDFPN, there is a simple expression for χ:

$$\chi = \max_{C \in \Gamma} T_C / N_C, \tag{5.3}$$

where C is any circuit in Γ, T_C is the sum of the holding times in circuit C, N_C is the number of tokens in the initial marking of C (see [23] for a proof).

To the best of the authors' knowledge, the derivation of a general closed form expression for the constant χ is an open problem (see [20]).

The remainder of this section focuses on the derivation of such properties as monotonicity, concavity and decomposability of the constant χ, which can be useful in numerical computations.

5.1 Bounds

Owing to the convergence result of (5.1), stochastic monotonicity properties obtained in §3 admit the following immediate corollaries:

Corollary 5.1 *Under the assumptions of Proposition 3.1 (monotonicity with respect to the holding times in $\leq_{icx}$ sense),*

$$\chi \leq \tilde{\chi}, \tag{5.4}$$

where $\tilde{\chi}$ is the cycle time associated with $\tilde{\Gamma}$. In particular, taking $\tilde{\alpha}(k) = E[\alpha(k)]$ and using (5.3) yields

$$\chi \geq \max_{C \in \Gamma} E[T_C] / N_C. \tag{5.5}$$

Corollary 5.2 *Under the assumptions of Proposition 3.3 (monotonicity with respect to the initial marking),*

$$\chi \geq \chi', \tag{5.6}$$

where χ' is the cycle time associated with Γ'.

Corollary 5.3 *Under the assumptions of Proposition 3.4 (monotonicity with respect to the topology),*

$$\chi \leq \bar{\chi}, \tag{5.7}$$

where $\bar{\chi}$ is the cycle time associated with $\bar{\Gamma}$.

Remark 5.1 Following Remark 4.2, one obtains that the throughput in general Fork/Join queueing networks with blocking analysed in Ammar and Gershwin [3] and Dallery, Liu and Towsley [16] is stochastically monotone (decreasing) in service times and (increasing) in the buffer sizes and in the customer populations. This generalizes the monotonicity results of throughput in

- cyclic queueing networks with finite buffers by Shanthikumar and Yao [26].
- assembly networks by Adan and van der Wal [1].

Remark 5.2 Following Remarks 3.5 and 3.6, computable upper and lower bounds for the throughput in synchronized queueing networks (cf. [9]) and in Fork/Join queueing networks with blocking (cf. [3]) can be obtained by adding and removing synchronization or blocking constraints, respectively.

Remark 5.3 Similarly, inserting additional stations at the beginning or at the end of a production line yields an upper bound of the throughput.

Remark 5.4 In parallel to Corollaries 5.1 and 5.2, the monotonicity of throughput in closed queueing networks with Bernoulli routing and infinite buffers has been obtained in the literature. Shanthikumar and Yao [25] showed that in some product-form queueing networks, the throughput is increasing in service rates. Tsoucas and Walrand [30] and Adan and van der Wal [2] proved that in some non-product-form queueing networks, the throughput is increasing in the customer population and the number of servers.

5.2 Concavity with respect to the Initial Marking

The concavity properties established in §4 for counters together with relation (5.2) readily imply a related concavity property of the throughput with respect to the initial marking.

Corollary 5.4 *If the holding times are mutually independent and if the holding times of transition j are i.i.d. with PERT distribution, then, χ^{-1} is increasing and concave in the initial marking $\mu \in I\!N^{|\mathcal{E}|}$.*

Remark 5.5 Following Remarks 4.2 and 4.5, one obtains for instance that the throughput in Fork/Join queueing networks with blocking and with PERT type service times is stochastically monotone (decreasing) in the service times and (increasing) and concave in the buffer sizes and the customer populations. All this generalizes the monotonicity and concavity results of the throughput in the following queueing systems with exponential service times:

- cyclic queueing networks with finite buffers by Shanthikumar and Yao [26].
- cyclic queueing networks with infinite buffers by Shanthikumar and Yao [27].
- tandem queueing networks with finite buffers by Meester and Shanthikumar [21] and Anantharam and Tsoucas [4].

5.3 Decomposition Property

Let g be the number of the maximal strongly connected subgraphs in Γ (see Appendix B for definitions). Define the *reduced graph* of Γ, denoted $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$, to be the graph with vertices $\mathcal{V}^* = \{1, 2, \ldots, g\}$ and with edges

$$\mathcal{E}^* = \{(h,k) | 1 \leq h < k \leq g,\ \exists j, j' \in \mathcal{T},\ j \in \Gamma_h,\ j' \in \Gamma_k \text{ and there is a path from } j \text{ to } j' \text{ in } \mathcal{E}\}$$

The reduced graph describes the precedence relations between the maximal strongly connected subgraphs of the SDFPN. Obviously, $\mathcal{G}$ is acyclic. Without loss of generality, it will be assumed that the numbering of the vertices is compatible with the graph in the sense that $(i,j) \in \mathcal{E}^*$ implies $i < j$. The root vertices of $\mathcal{G}^*$ will be numbered $1, \ldots, g_0 \in \mathcal{V}$, and the set of predecessors of h in $\mathcal{G}^*$ by $\pi^*(h)$.

Let $\Gamma_1 = (\mathcal{T}_1, \mathcal{E}_1)$, ..., $\Gamma_g = (\mathcal{T}_g, \mathcal{E}_g)$ be the set of all these subgraphs. It is easy to prove that the preceding set of subgraphs is uniquely defined and that these subgraphs have no vertices in common. A decision free Petri net can be associated with each subgraph. The initial marking in Γ_h, $1 \leq h \leq g$, is the restriction of the initial marking to the places of $\mathcal{P}_h$, and the holding times in Γ_h are those of the initial Petri net. Similarly, define Γ^h, $1 \leq h \leq g$, to be the SDFPN obtained by taking all the transitions in Γ_h and those of Γ from which there are paths to the transitions of Γ_h.

For $1 \leq h \leq g$, define χ_h and χ^h to be the asymptotic cycle times of Γ_h and Γ^h, respectively. The quantity χ_h is also called the *proper cycle time* of Γ_h in the sense that it is defined on Γ_h alone. It follows from (5.1) and the fact that Γ_h, $1 \leq h \leq g$, are strongly connected that

$$\chi^h = \lim_{k \to \infty} X_j(k)/k = \lim_{k \to \infty} E[X_j(k)]/k \quad a.s., \qquad 1 \leq h \leq g, \quad j \in \Gamma_h, \tag{5.8}$$

$$\chi^h = \chi_h, \qquad h = 1, \ldots, g_0, \tag{5.9}$$

$$\chi^h \geq \chi_h, \qquad h = g_0 + 1, \ldots, g, \tag{5.10}$$

$$\chi = \max_{1 \leq h \leq g} \chi^h. \tag{5.11}$$

Proposition 5.5

$$\chi = \max_{1 \leq h \leq g} \chi_h. \tag{5.12}$$

In words, the cycle time of a SDFPN is determined by its sub-SDFPN with the largest cycle time. This property indicates that the computation of the cycle time or throughput of a general SDFPN can be reduced to that of its strongly connected subgraphs. The proof of Proposition 5.5 is forwarded to Appendix B.

The decomposition property was first established by the authors [10] for synchronized queueing networks. The proof in Appendix B extends that of [10]. Note that our proof does not use the fact that the SDFPN is recycled. In fact, the result holds for general decision free Petri nets provided that a firing occurs as soon as possible.

As an application of the decomposition property to the comparison of cycle times, we take two SDFPN's Γ and Γ' with possibly completely different structures. Let them be decomposed into g and g' subnetworks $\Gamma_1, \ldots, \Gamma_g$ and $\Gamma'_1, \ldots, \Gamma'_{g'}$, respectively. Denote by $\chi_1, \ldots, \chi_g$ and $\chi'_1, \ldots, \chi'_{g'}$ the corresponding cycle times. If one can prove, by using the comparison results of §5.1, that

$$\forall i,\ 1 \leq i \leq g, \quad \exists j,\ 1 \leq j \leq g' : \qquad \chi_i \leq \chi'_j,$$

then the cycle times of Γ and Γ', referred to as χ and χ', respectively, satisfy the relation

$$\chi \leq \chi'.$$

6 Marking Distribution

The marking in certain places also exhibits interesting stochastic ordering properties. Roughly speaking, the places in question are those that do not belong to a strongly connected component of the SDFPN, which are the only places the marking of which is not structurally bounded. The properties of interest are established through simple examples.

6.1 Evolution Equation

Assume that Γ has several strongly connected subgraphs, as defined in the previous section, and let Γ_h be one of them, and assume that h is not an source vertex in the reduced graph. Assume for sake of simplicity that each transition j of Γ_h is connected to a single transition $p(j)$ of $\{\Gamma_l,\ l < h\}$ and that $\mu(p(j), j) = 0$. Then the evolution equation (3.1) implies

$$\begin{aligned} X_j(k) &= \max\left(\max_{\{i\in\pi(j)\cap\Gamma_h\}}(X_i(k-\mu(i,j))+\alpha_i(k-\mu(i,j))), \left(X_{p(j)}(k))+\alpha_{p(j)}(k)\right)\right), \\ & k = 1,2,\ldots, \end{aligned}$$

for all j in Γ_h. Therefore, the differences

$$W_j(k) = X_j(k) - X_{p(j)}(k) - \alpha_{p(j)}(k), \quad k = 1,2,\ldots, \quad j \in \Gamma_h \tag{6.1}$$

are easily seen to satisfy the evolution equation

$$\begin{aligned} W_j(k) &= \max\left(\max_{\{i\in\pi(j)\cap\Gamma_h\}}(W_i(k-\mu(i,j))+\alpha_i(k-\mu(i,j))-\beta_{i,j}(k)), 0\right), \\ & k = 1,2,\ldots, \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} \beta_{i,j}(k) &= X_{p(j)}(k)+\alpha_{p(j)}(k) - X_{p(i)}(k-\mu(i,j)) - \alpha_{p(i)}(k-\mu(i,j)), \\ & k = 1,2,\ldots, \quad i,j \in \Gamma_h. \end{aligned}$$

Observe that $W_j(k)$ can be interpreted as the *response time* (i.e., the difference of the departure and the arrival times) of the k-th token in place $(p(j), j)$. A similar equation can be derived in the general case (see [5]).

6.2 Stochastic Ordering Results

The evolution equation (6.2) falls in the class of considered in §3.D.1 of [12]. Therefore, $W_j(k)$ is stochastically increasing and convex in the variables α and β. Interesting applications of this property arise when the holding times are all mutually independent, so that the random variables $\{\alpha_i(k)\}$, $i \in \Gamma_h$ and $\{\beta i, j(k)\}$, $i, j \in \Gamma_h$ are also mutually independent.

For example, applying the results of §3.D.1 in [12] when projecting onto the sigma-field generated by the random variables $\{\alpha_i(k)\}$, $i \in \Gamma_h$, we get that

$$\{W(k)\} \geq_{icx} \{\widetilde{W}(k)\}, \tag{6.3}$$

where $\widetilde{W}(k)$ satisfies the recursion

$$\begin{aligned}\widetilde{W}_j(k) &= \max\left(\max_{\{i\in\pi(j)\cap\Gamma_h\}}\left(\widetilde{W}_i(k-\mu(i,j))+\alpha_i(k-\mu(i,j))-E[\beta_{i,j}(k)]\right),0\right),\\ & \quad k=1,2,\ldots,\end{aligned} \tag{6.4}$$

This evolution equation is simpler than the earlier one in that the influence of the network $\{\Gamma_l,\ l<h\}$ on Γ_h is captured by the first moments of the variables β only.

The conditions under which a stationary solution of (6.2) and (6.4) and a stationary marking exist are given in [5,11]. Under these conditions, one can apply Little's formula, to deduce from (6.3) the following bound on M_j, the stationary marking in place $(p(j),j)$:

$$E[M_j]=E[W_j]\lambda_{p(j)}\geq E[\widetilde{W}_j]\lambda_{p(j)}, \tag{6.5}$$

where W_j and $\widetilde{W}_j$ respectively denote the stationary solutions of (6.2) and (6.4) and where λ_j denotes the throughput of transition j.

Similarly, under the preceding independence assumptions, it is easily seen that the variables $W_j(k)$, $j\in\mathcal{T}_h$, are stochastically increasing and convex in the holding times of Γ_h. More generally, results analogous to those of §3, namely, the association of the response times and their stochastic and convex monotonicity with respect to the holding time sequences and to the initial marking of Γ_h, can be derived in the very same way.

Remark 6.1 The preceding convex monotonicity of the response times with respect to the arrival times and the holding times extends the validity of Ross' conjecture (cf. Remark 3.3) and generalizes a folk theorem (cf. Hajek [18]) for $G/GI/1$ queues to the response times in SDFPNs.

A Proof of the Log Concavity of the PERT Type Distribution Functions

Let $G=(V,E)$ be a stochastic PERT graph. For all $v\in V$, denote by X_v the weight of v, and by f_v and F_v its density and its distribution functions, respectively. Denote by F_G the distribution function of the weight of the critical path of G.

Theorem A.1 *Assume that the random variables X_v are mutually independent. If for all $v\in V$, either X_v is deterministic or its density function f_v is log concave, then F_G is log concave.*

Corollary A.2 *The PERT type distribution functions are log concave.*

In order to prove the theorem, we need the following lemmas.

Lemma A.3 (Prekopa [22]) *Let h be a log concave function defined on $I\!R^m \times I\!R^n$. Then the function defined on $I\!R^n$ by*

$$g(y) = \int_{I\!R^m} h(x, y)dx$$

is log concave, provided $g(y) < \infty$ for $y \in I\!R^n$.

Proof. See Prekopa [22] or Eaton [17, Theorem 4.8, p. 79]. ∎

Lemma A.4 *Suppose h_1 and h_2 are log concave functions on $I\!R^m$ and on $I\!R^n$, respectively. Let $p : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, m\}$ be an arbitrary mapping. For all $x \in I\!R^m$, let $x_p = (x_{p(1)}, \ldots, x_{p(n)}) \in I\!R^n$. Then the function*

$$h(y) = \int_{I\!R^m} h_1(y - x_p)h_2(x)dx$$

is log concave, provided $h(y) < \infty$ for $y \in I\!R^n$.

Proof. Let $\phi_1(x, y)$ and $\phi_2(x, y)$ be functions defined on $I\!R^{m \times n}$ by

$$\phi_1(x, y) = h_1(y - x_p), \qquad \phi_2(x, y) = h_2(x), \qquad x \in I\!R^m, \quad y \in I\!R^n.$$

For all $0 < \alpha < 1$, $(x, y), (x', y') \in I\!R^{m \times n}$,

$$\begin{aligned} \phi_1(\alpha x + (1 - \alpha)x', \alpha y + (1 - \alpha)y') &= h_1(\alpha y - \alpha x_p + (1 - \alpha)y' - (1 - \alpha)x'_p) \\ &\geq h_1^{\alpha}(y - x_p)h_1^{(1-\alpha)}(y' - x'_p) \\ &= \phi_1^{\alpha}(x, y)\phi_1^{(1-\alpha)}(x', y'), \end{aligned}$$

which implies that $\phi_1(x, y)$ is log concave on $I\!R^{m \times n}$. As $h_2(x)$ is log concave on $I\!R^m$, $\phi_2(x, y)$ is log concave on $I\!R^{m \times n}$.

Therefore, the function $h_1(y - x_p)h_2(x)$ is log concave on $I\!R^{m \times n}$. Applying Lemma A.3 readily yields the log concavity of $h(y)$. ∎

Lemma A.5 *Let $h(x_1, x_2, \ldots, x_n)$ be the joint distribution function of the RV's $X_1, X_2, \ldots, X_n$. Let $c_1, c_2, \ldots, c_n$ be n arbitrary constants, and let $Y_i = X_i + c_i$, $1 \leq i \leq n$. If $h(x_1, x_2, \ldots, x_n)$ is log concave on $I\!R^n$, then the distribution function*

$$g(y_1, y_2, \ldots, y_n) = P[Y_1 \leq y_1, Y_2 \leq y_2, \ldots, Y_n \leq y_n]$$

is also log concave on $I\!R^n$.

Proof. Let $c = (c_1, c_2, \ldots, c_n)$. It is easy to see that for all $y \in I\!R^n$, $g(y) = h(y - c)$. Thus, for all $0 < \alpha < 1$, $y = (y_1, \ldots, y_n), y' = (y'_1, \ldots, y'_n) \in I\!R^n$,

$$\begin{aligned} & g(\alpha y + (1-\alpha)y') \\ &= h(\alpha(y-c) + (1-\alpha)(y'-c)) \\ &\geq h^{\alpha}(y-c)h^{(1-\alpha)}(y'-c) \\ &= g^{\alpha}(y)g^{(1-\alpha)}(y') \end{aligned}$$

Therefore, $g(y)$ is log concave on $I\!R^n$. ∎

Lemma A.6 *Let $h(x_1, x_2, \ldots, x_n)$ be the joint distribution function of the RV's $X_1, X_2, \ldots, X_n$. For $1 \leq i \leq m$, let S_i be a subset of $\{1, 2, \ldots, n\}$, and let*

$$Y_i = \max_{j \in S_i} X_j.$$

If $h(x_1, x_2, \ldots, x_n)$ is log concave on $I\!R^n$, then the distribution function

$$g(y_1, y_2, \ldots, y_m) = P[Y_1 \leq y_1, Y_2 \leq y_2, \ldots, Y_m \leq y_m]$$

is log concave on $I\!R^m$.

Proof.

$$\begin{aligned} & P[Y_1 \leq y_1, Y_2 \leq y_2, \ldots, Y_m \leq y_m] \\ &= P[\max_{j \in S_1} X_1 \leq y_1, \max_{j \in S_2} X_2 \leq y_2, \ldots, \max_{j \in S_m} X_m \leq y_m] \\ &= P[X_1 \leq \min_{i \in T_1} y_i, X_2 \leq \min_{i \in T_2} y_i, \ldots, X_n \leq \min_{i \in T_n} y_i], \end{aligned}$$

where $T_i = \{j | i \in S_j\}$, $1 \leq i \leq n$. Thus,

$$g(y_1, y_2, \ldots, y_m) = h\left(\min_{i \in T_1} y_i, \min_{i \in T_2} y_i, \ldots, \min_{i \in T_n} y_i\right).$$

Using the facts that h is increasing (as h is a distribution function) and that h is log concave, we obtain that for all $0 < \alpha < 1$, $y = (y_1, \ldots, y_m), y' = (y'_1, \ldots, y'_m) \in I\!R^m$,

$$\begin{aligned}
g(\alpha y + (1-\alpha)y') &= h\left(\min_{i \in T_1}(\alpha y_i + (1-\alpha)y'_i), \ldots, \min_{i \in T_n}(\alpha y_i + (1-\alpha)y'_i)\right) \\
&\geq h\left(\alpha(\min_{i \in T_1} y_i) + (1-\alpha)(\min_{i \in T_1} y'_i), \ldots, \alpha(\min_{i \in T_n} y_i) + (1-\alpha)(\min_{i \in T_n} y'_i)\right) \\
&\geq h^{\alpha}\left(\min_{i \in T_1} y_i, \ldots, \min_{i \in T_n} y_i\right) h^{(1-\alpha)}\left(\min_{i \in T_1} y'_i, \ldots, \min_{i \in T_n} y'_i\right) \\
&= g^{\alpha}(y) g^{(1-\alpha)}(y').
\end{aligned}$$

Therefore, $g(y)$ is log concave on $I\!R^m$. ∎

Proof of Theorem A.1.

For all $v \in V$, let $\pi(v)$ and $\sigma(v)$ denote the sets of predecessors and successors of v, respectively. Define the level of vertex $v \in V$, denoted by $l(v)$ as follows:

$$\begin{aligned}
l(v) &= 1, \qquad \pi(v) = \emptyset; \\
l(v) &= \max_{u \in \pi(v)} l(u) + 1.
\end{aligned}$$

For sake of simplicity, we assume, without loss of generality, that for all $u, v \in V$, $l(v) = l(u) + 1$ if $u \in \pi(v)$. This can be achieved by inserting "dumy vertices" in G with zero weight.

Let $H = \max_{v \in V} l(v)$ be the height of G, and w_i, $1 \leq i \leq H$, the width of G at level i, defined as

$$w_i = \sum_{v \in V} \mathbf{1}_{\{l(v) = i\}}.$$

Let r_i be the number of vertices at level i with random weight. We relabel the vertices of V in such a way that the vertices at level i are labeled $v_{i,1}, v_{i,2}, \ldots, v_{i,r_i}, v_{i,r_i+1}, \ldots, v_{i,w_i}$, where $v_{i,r_i+1}, \ldots, v_{i,w_i}$ are the vertices with deterministic weights. For $1 \leq i \leq H$, let $h_i : I\!R^{r_i} \to I\!R^+$ be the joint density function of the weights of the vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,r_i}$:

$$h_i(x_{i,1}, \ldots, x_{i,r_i}) = \prod_{j=1}^{r_i} f_{v_{i,j}}(x_{v_{i,j}}).$$

The paths of G that start with some vertex of level 1 and terminate with some vertex of level H are labeled $1, 2, \ldots, P$, where P is the total number of such paths. Denote by

$p_i : \{1, 2, \ldots, P\} \rightarrow \{1, 2, \ldots, w_i\}$, $1 \leq i \leq H$, the mapping such that $p_i(n) = k$ if and only if $v_{i,k}$ is on the n-th path.

For $1 \leq i \leq H$, let $g_i(y_1, \ldots, y_P)$ be the joint distribution function of the path weights truncated at level i, i.e., the weights of the paths from level 1 to (including) level i. Let $g_i'(y_1, \ldots, y_P)$ be the joint distribution function of the path weights truncated at level i excluding the deterministic weights at level i, i.e., the weights of the paths from level 1 to (including) level $i-1$ and the weight of level i if it has a density function. Let $g_0(y_1, \ldots, y_P)$ be defined by

$$\begin{aligned} g_0(y) &= 1, \qquad y \in I\!R^{+P}, \\ g_0(y) &= 0, \qquad \text{otherwise.} \end{aligned}$$

It then follows that for all $1 \leq i \leq H$,

$$\begin{aligned} g_i'(y_1, \ldots, y_P) &= \int_{I\!R^{r_i}} g_{i-1}(y_1 - x_{i,p_i(1)}, \ldots, y_P - x_{i,p_i(P)}) h_i(x_{i,1}, \ldots, x_{i,r_i}) dx_{i,1} \ldots dx_{i,r_i}, \\ g_i(y_1, \ldots, y_P) &= g_i'(y_1 - c_{i,p_i(1)}, \ldots, y_P - c_{i,p_i(P)}), \end{aligned}$$

where $c_{i,1} = \ldots = c_{i,r_i} = 0$, and $c_{i,r_i+1}, \ldots, c_{i,w_i}$ are the weights of the vertices $v_{i,r_i+1}, \ldots, v_{i,w_i}$.

It is easy to see that the functions $g_0(y_1, \ldots, y_P)$ and $h_i(x_{i,1}, \ldots, x_{i,w_i})$, $1 \leq i \leq H$, are log concave functions (the latter are in fact the products of log concave functions). Using Lemmas A.4 and A.5, one shows by induction on i, $1 \leq i \leq H$, that $g_i'(y_1, \ldots, y_P)$ and $g_i(y_1, \ldots, y_P)$ are log concave on $I\!R^P$.

Since the critical weight of G is the maximum of the weights of all the paths $1, 2, \ldots, P$, we apply Lemma A.6 to $g_H(y_1, \ldots, y_P)$ which allows us to conclude that F_G is log concave on $I\!R$. ∎

B Proof of the Decomposition Property of the Cycle Time

Recall that a strongly connected graph is a directed graph in which the existence of a directed path from v_1 to v_2 implies the existence of another path from v_2 to v_1. A maximal strongly connected subgraph of a graph G is a strongly connected subgraph of G such that no other subgraph of G covering it is strongly connected.

Proof of Proposition 5.5. It follows from the relations (5.9—5.11) that

$$\chi \geq \max_{1 \leq h \leq g} \chi_h \tag{B.1}$$

We are now going to prove that the reversed inequality also holds.

For every Γ_h $(1 \le h \le g)$, we construct a new SDFPN $\bar{\Gamma}_h = (\bar{\mathcal{T}}_h, \bar{\mathcal{E}}_h)$ by adding a fictive transition t_0^h and a set of fictive places:

$$\begin{aligned}\bar{\mathcal{T}}_h &= \mathcal{T}_h \bigcup \{t_0^h\}, \\ \bar{\mathcal{E}}_h &= \mathcal{E}_h \bigcup \{(t_0^h \to t_0^h)\} \bigcup \left(\bigcup_{t \in T_h} \{(t_0^h \to t)\} \right)\end{aligned}$$

The firing time of t_0^h is zero, and the holding times of the fictive places are also zero. The initial marking of $(t_0^h \to t_0^h)$ is one while all other fictive places contain no initial token.

Let $\bar{\chi}_h$ be the cycle time of $\bar{\Gamma}_h$, $h = 1, \ldots, g$. Observe that $\bar{\Gamma}_h$ differs from Γ_h in that there is a fictive transition which constantly produces tokens and sends them to the fictive places preceding the transitions in T_h. Owing to the facts that the the firing of the fictive transition is instantaneous (i.e., the holding times are zero), such a difference does not change the firing epochs of the transitions in T_h. Therefore we obtain

$$\hat{\chi}_h = \bar{\chi}_h, \qquad 1 \le h \le g \tag{B.2}$$

Now let $\bar{\Gamma} = (\bar{\mathcal{T}}, \bar{\mathcal{E}})$ be defined as

$$\begin{aligned}\bar{\mathcal{T}} &= \bigcup_{h=1}^{g} \bar{\mathcal{T}}_h \\ \bar{\mathcal{E}} &= \mathcal{E} \bigcup \mathcal{E}' \bigcup \bigcup_{h=1}^{g} \bar{\mathcal{E}}_h\end{aligned}$$

where

$$\mathcal{E}' = \bigcup_{(h,k) \in \mathcal{E}^*} \bigcup_{t \in T_h} \{(t \to t_0^k)\}.$$

The places associated with $\mathcal{E}'$ have zero initial marking.

Intuitively, $\bar{\Gamma}$ is constructed in such a way that all the precedence relations which exist in Γ are preserved in $\bar{\Gamma}$, and that the fictive transitions and places affect only the pairs of transitions (s, t) with $s \in T_h$, $t \in T_k$, and $(h, k) \in \mathcal{E}^*$.

Denote by $\bar{\chi}$ the cycle time of $\bar{\Gamma}$. It then follows (cf. Proposition 3.4) that

$$\chi \le \bar{\chi} \tag{B.3}$$

Furthermore, from the construction of $\bar{\Gamma}$, we observe that the transition firings in $\bar{\Gamma}$ are pipelined in the sense that the transitions in SDFPNs $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_g$ are fired in the partial order defined by reduced graph $\mathcal{G}$. This implies that

$$\bar{\chi} = \max_{1 \leq h \leq g} \bar{\chi}_h \tag{B.4}$$

Relations (B.2,B.3,B.4) readily yield the following inequality:

$$\chi \leq \bar{\gamma} = \max_{1 \leq h \leq g} \bar{\chi}_h = \max_{1 \leq h \leq g} \chi_h \tag{B.5}$$

The assertion (5.12) is clearly a consequence of (B.1) and (B.5). ∎

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